Reliable Two-Dimensional Graphing Methods for Mathematical Formulae with Two Free Variables

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SIGGRAPH 2001
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An Equation of a Circle

$$x^2 + y^2 = 36$$

Example Inequality

$$888\sin(2x+y)\sin(2x-y)\sin(x+2y)\sin(x-2y) > y-x$$

Example Equation

$$\sin\sin\sqrt{\left|x^2\sin x+y^2\cos x\right|}+\sin x=\cos\sin\sqrt{\left|x^2\cos y+y^2\sin y\right|}+\cos y$$

Example Inequality

$$1.9\cos\left(69\theta + 2\left\lfloor \frac{27r}{\pi} \right\rfloor\right) < \frac{2}{\pi}\operatorname{mod}(r, \pi) + \sin\left(\theta + \pi\left\lfloor \frac{r}{\pi} \right\rfloor\right) - 1$$

$$r < 3\pi$$

Example Equation

$$k = \left(\left\lfloor \frac{|x|}{4} \right\rfloor + 3 \right) m$$

$$j = \left\lfloor \frac{k}{2\pi} \operatorname{angle}\left(\operatorname{mod}(x,4) - 2, \operatorname{mod}(y,4) - 2 \right) \right\rfloor$$

$$\left(\operatorname{mod}(x,4) - 2 - d\cos\frac{2\pi j}{k} \right)\left(D\sin\frac{2\pi(j+1)}{k} - d\sin\frac{2\pi j}{k} \right) = \left(D\cos\frac{2\pi(j+1)}{k} - d\cos\frac{2\pi j}{k} \right)\left(\operatorname{mod}(y,4) - 2 - d\sin\frac{2\pi j}{k} \right)$$

$$\left| \operatorname{mod}(x,4) - 2 \right| + \left| \operatorname{mod}(y,4) - 2 \right| > \frac{3}{5}$$

$$d = \frac{5}{3} - \operatorname{signum}\left(\gcd(j+1, m) - 1 \right)$$

$$D = \frac{5}{3} - \operatorname{signum}\left(\gcd(j+2, m) - 1 \right)$$

$$m = 1 + \left\lfloor \frac{|y|}{4} \right\rfloor$$

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Example Inequality

$$\left|\cos\left(4\cos\min\left(\sin x+y,\,x+\sin y\right)\right)-\cos\left(4\sin\max\left(\sin y+x,\,y+\sin x\right)\right)\right|$$

Example Equation

$$\sin\left(2^{\lfloor y \rfloor} x \pm \frac{\pi}{4}\left(y - \lfloor y \rfloor\right) - \frac{\pi}{2}\right) = 0$$

Example Inequality

$$\sin\sin[x[\sin y-\cos x]] < \cos\cos[y[\cos x-\sin y]]$$

Example Inequality

$$\sin(\min([x+y]\sin[y-x],[y-x]\sin[x+y]))$$

$$)+\frac{1}{6400000}\left|8x^2+4(y-3)^2\right|$$

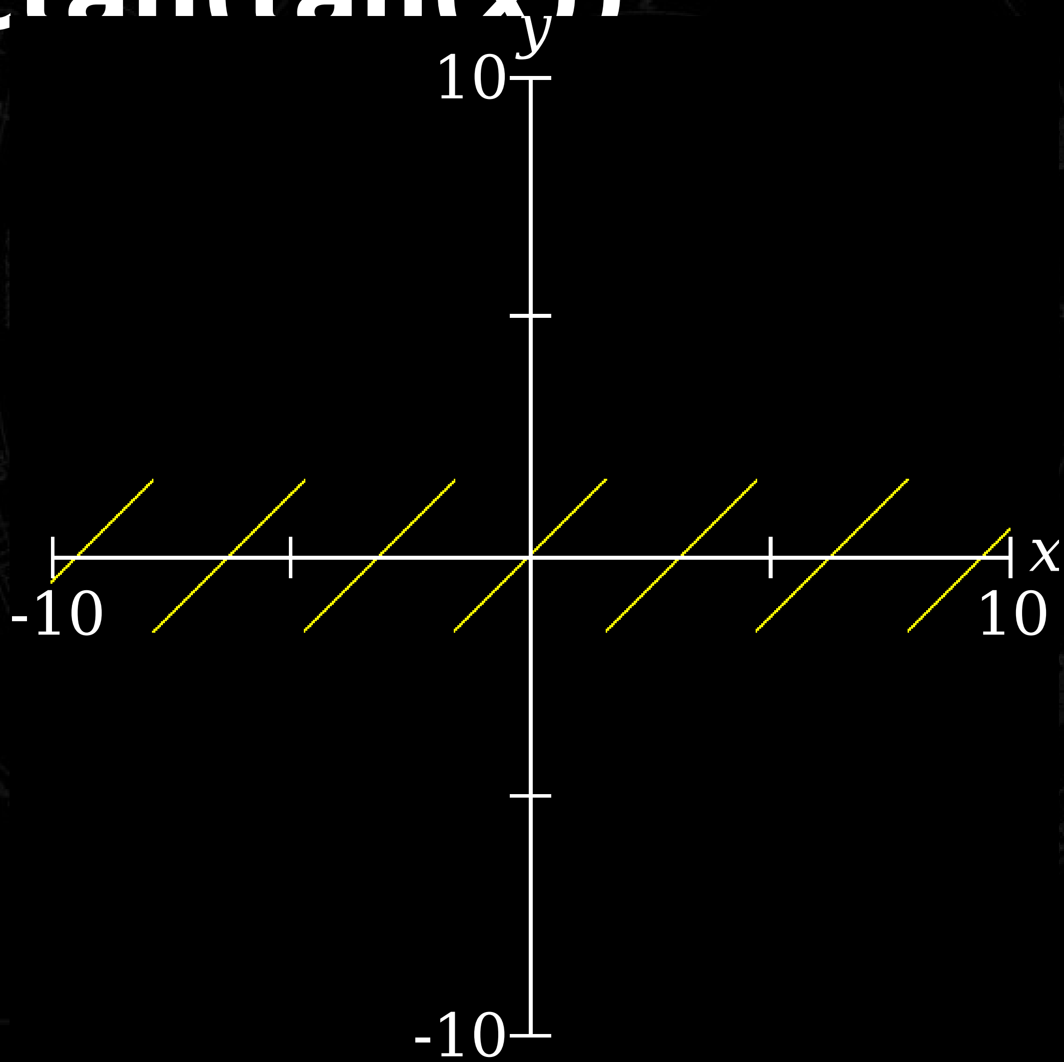
$$+\frac{1}{15}[6-y]$$

Isn't this already solved?

There are many utilities for doing this:

- **Computer Algebra Systems**
  - Mathematica, Maple, ...
- **Graphing Calculators**
  - Hewlett-Packard, Texas Instruments, ...
- **Graphing Software**
  - Curvus Pro, IAsolve, GraphingCalculator, ...

Correct Graph of *y*=Arctan(tan(*x*))



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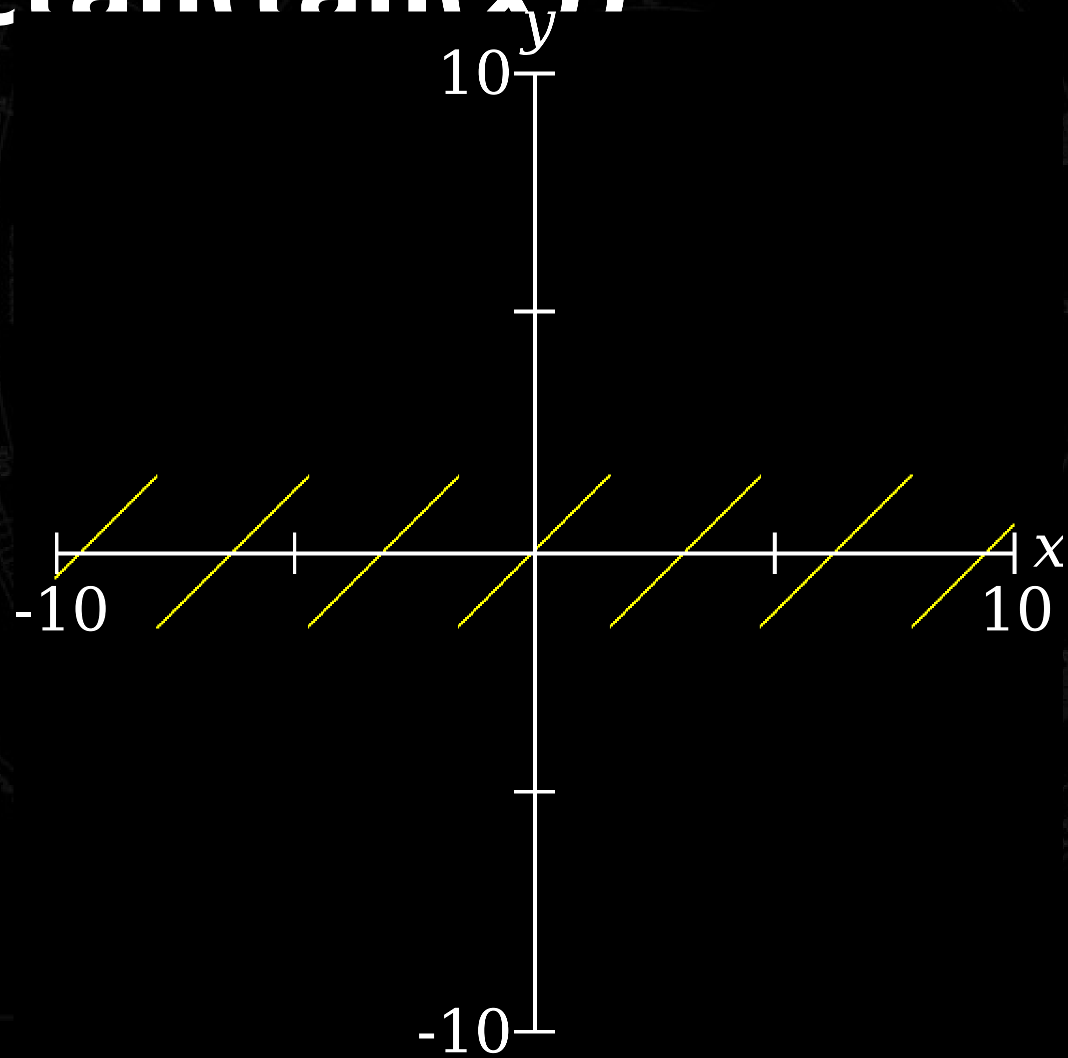

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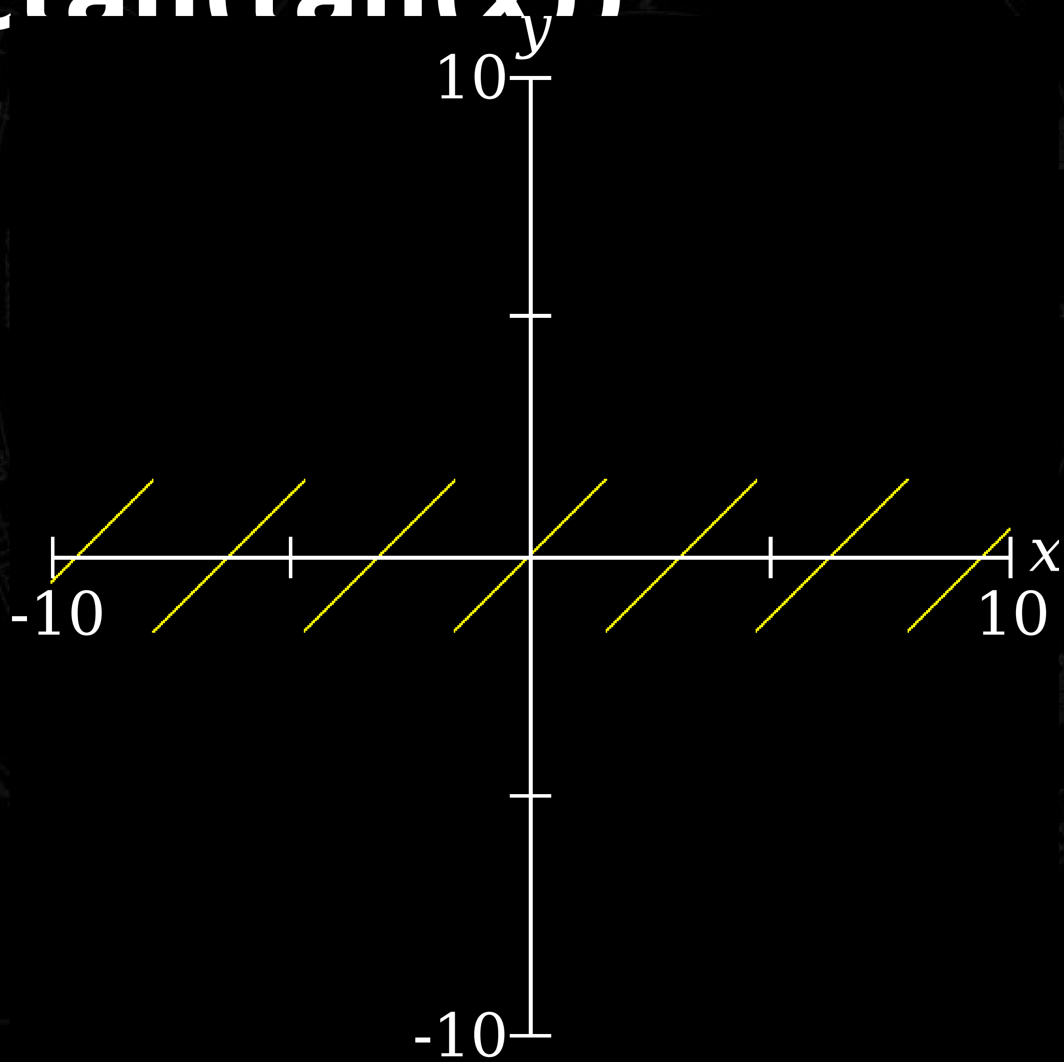

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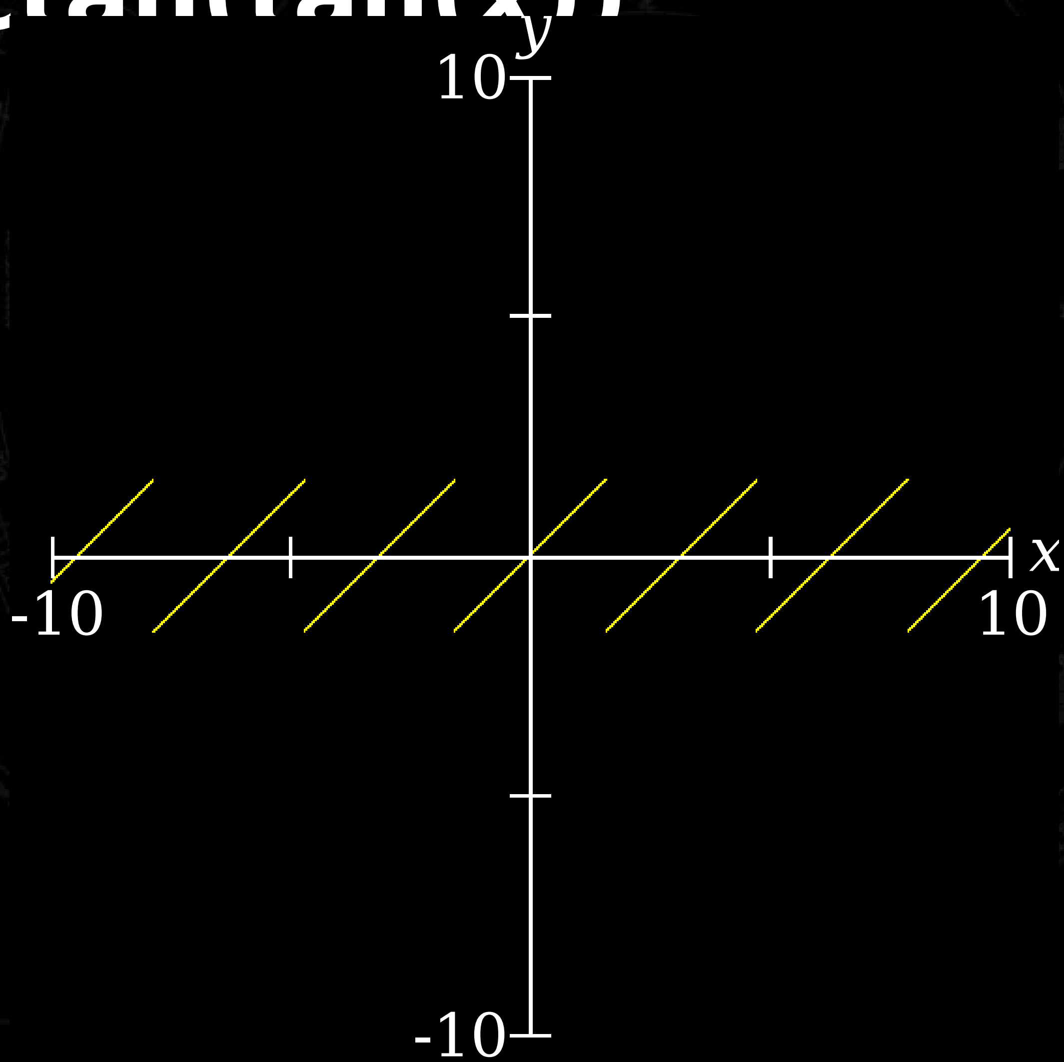

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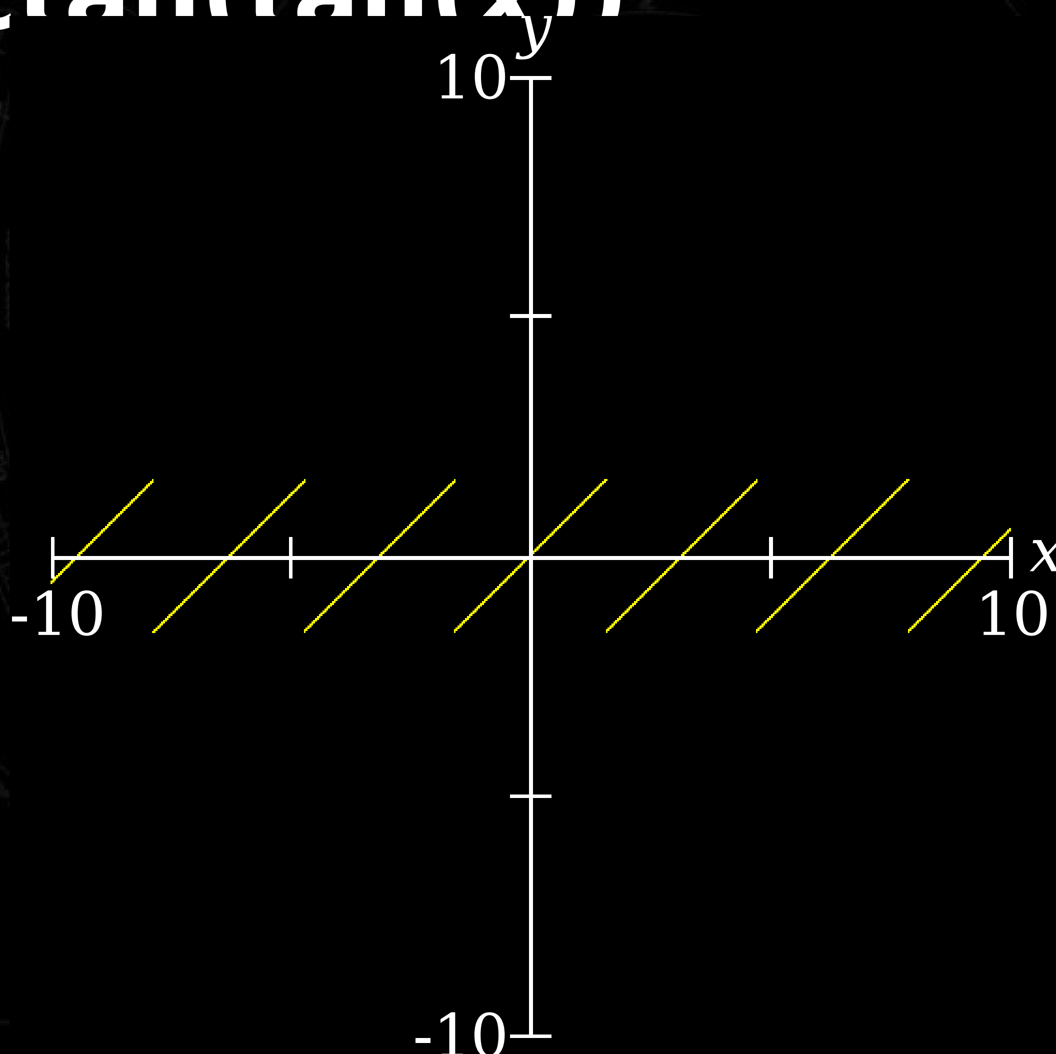

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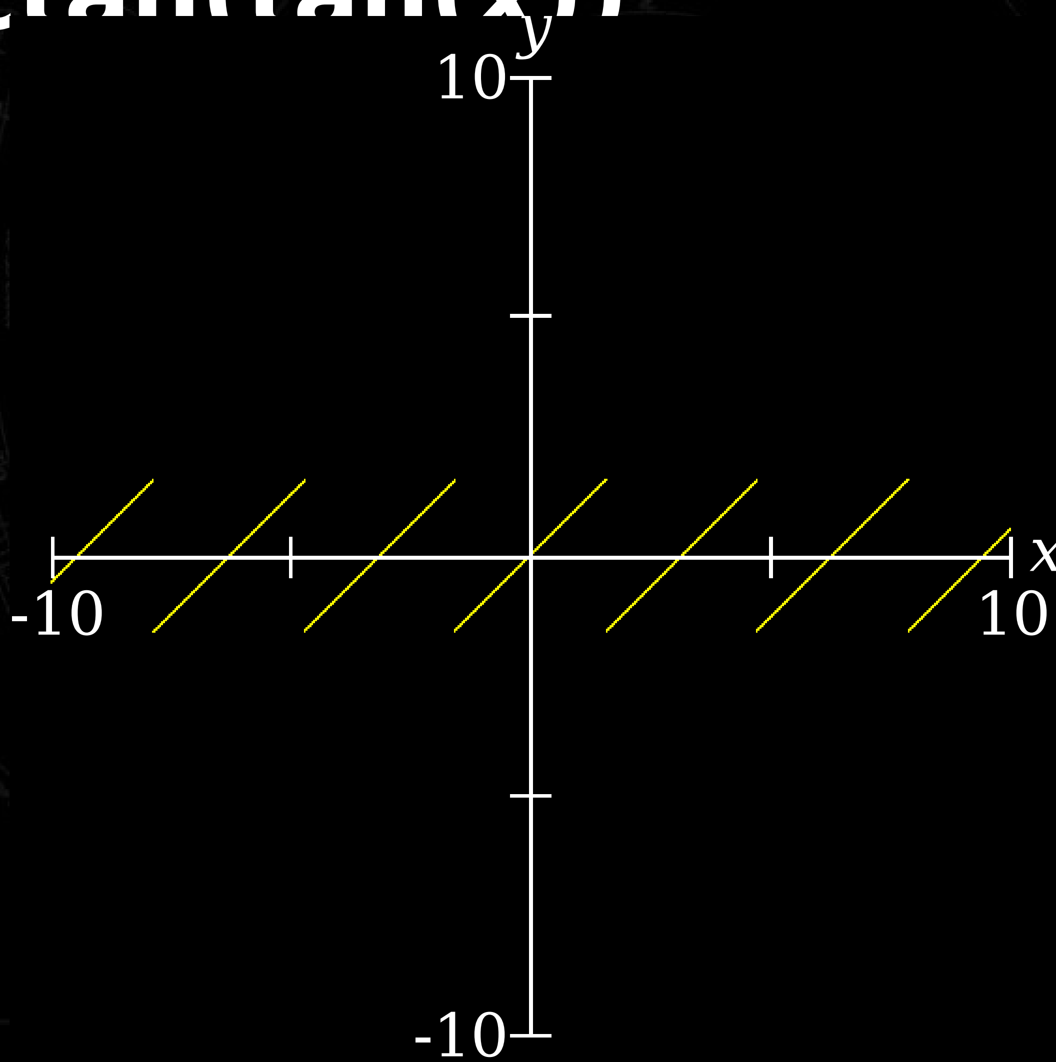
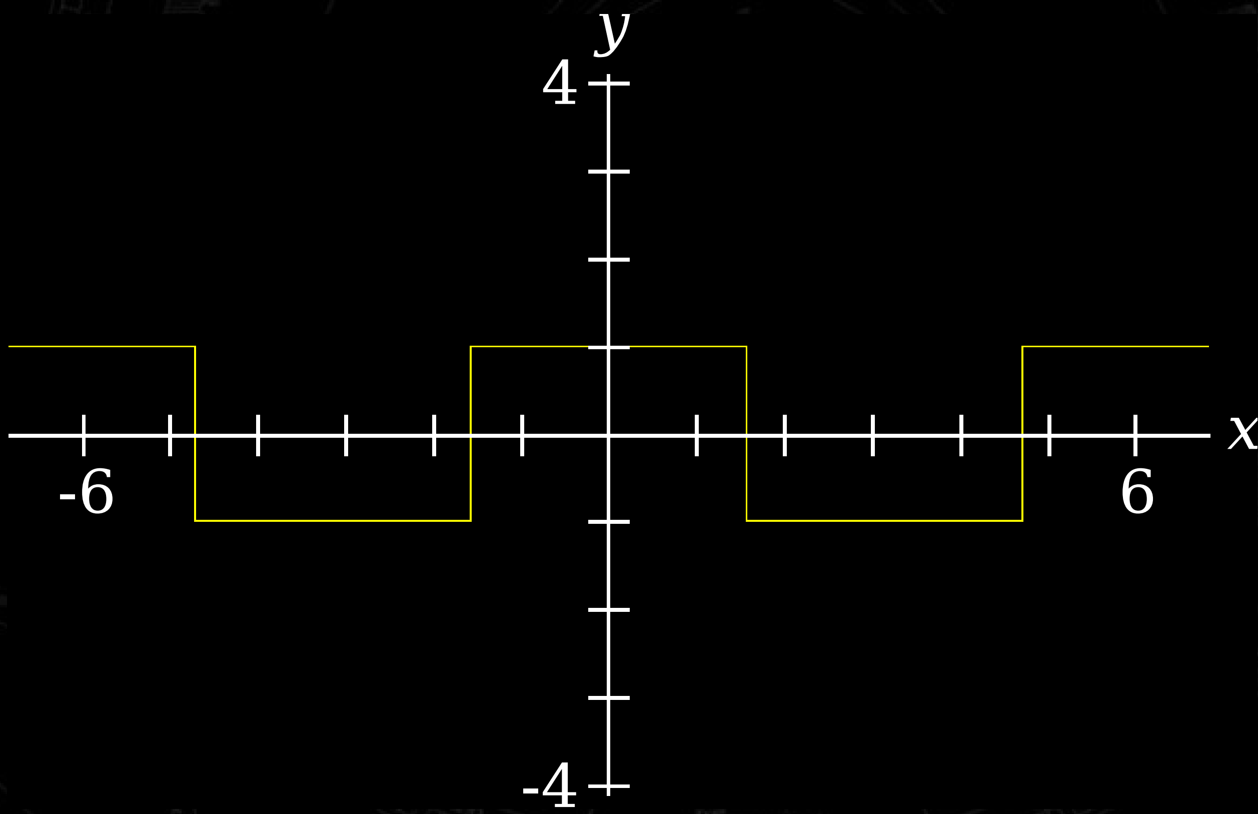


Correct Graph of $y = \sqrt[9^9]{\cos x}$

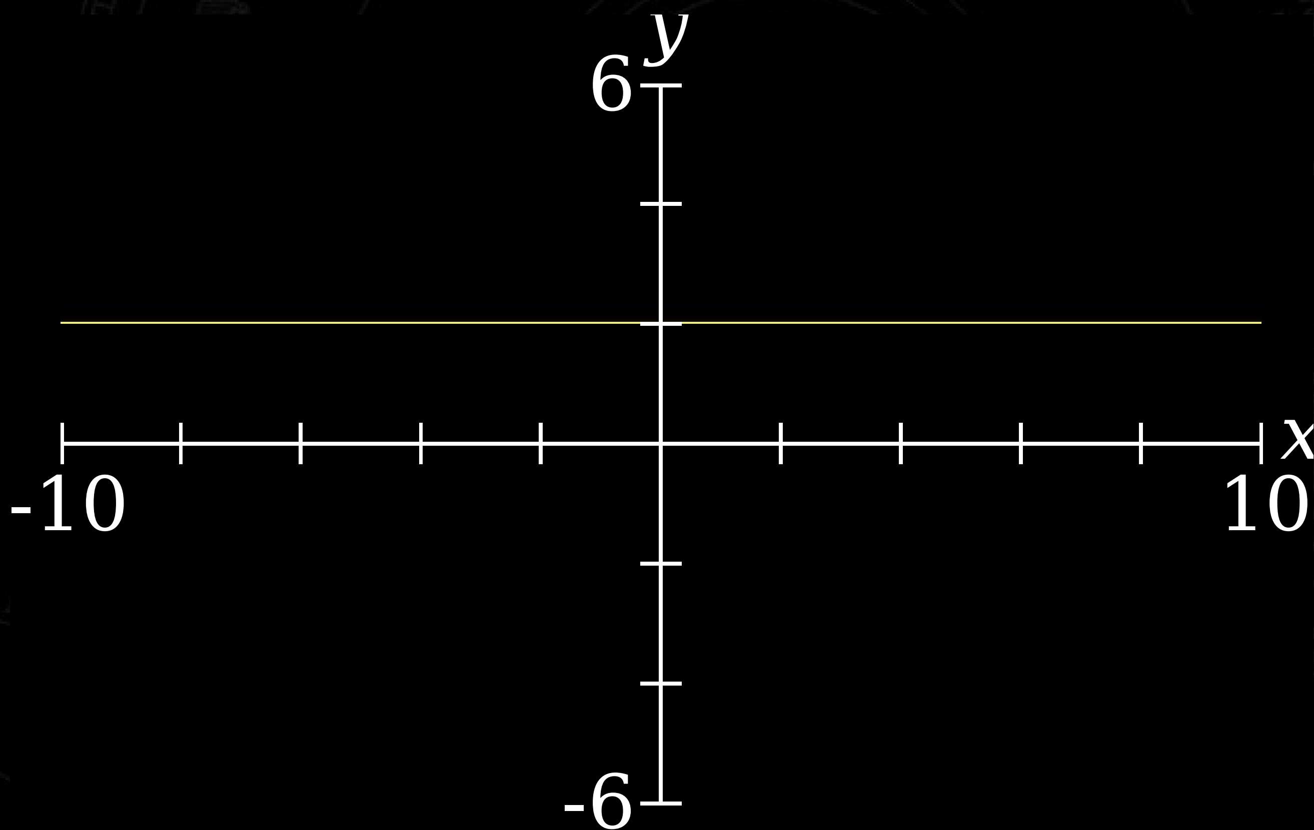


Floating-Point Arithmetic

- Many graphing programs use floating-point arithmetic to evaluate formulae
- Floating-point arithmetic is not exact

$$1.000 \div 3.000 \rightarrow 0.3333$$
$$0.3333 \times 3.000 \rightarrow 0.9999$$

Correct Graph of $y=(2^{55}+2+\sin x-2^{55})-\sin x$



Connect-the-Dots Graphing

Problems:

- Not all dots should be connected
- Dots may be far from the curve

Connect-the-Dots Graphing

Fundamental Problem:

- We haven't defined the graph's semantics

Graph Semantics

A pixel is:

Yellow ⇔ Solution Exists

Black ⇔ No Solution Exists

Example Equation

Example Inequality

$$\sin 30\sqrt{\left(\sin\frac{x-4}{3}\right)^2+\left(\cos\frac{y+6}{3}\right)^2}+$$

$$-\sin 20\sqrt{\left(\sin\frac{x-3}{2}\right)^2+\left(\cos\frac{y+5}{2}\right)^2}>0$$

Example Inequality

$$\lfloor \gcd(x,y) \rfloor + \left(\gcd(\lfloor x \rfloor, \lfloor y \rfloor) - 2 \right) \geq 0$$

Unfortunate Reality

- This naïve goal is impossible since graphing, as formalized, is not computable

Practical Graph Semantics

A pixel is:

Yellow ⇒ Solution Exists

Black ⇒ No Solution Exists

Red ⇒ Maybe, Maybe Not

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Reliable Graphing

- We now have a well-defined problem
- But how do we evaluate formulae?

Formula Evaluation

- Use interval arithmetic to evaluate formulae
- Interval arithmetic provides guaranteed bounds on accuracy

Interval Arithmetic

- **Compute using lower and upper bounds**

$$\langle 1.00, 1.00\rangle \div \langle 3.00, 3.00\rangle \rightarrow \langle 0.333, 0.334\rangle$$

$$\langle 0.333, 0.334\rangle \times \langle 3.00, 3.00\rangle \rightarrow \langle 0.999, 1.01\rangle$$

Interval Comparisons

$x+y^2 \in [2.13, 2.15]$

$y \in [2.16, 2.18]$

Is $x+y^2 < y$? Yes.

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$x+y^2 \in [2.13, 2.16]$

$y \in [2.15, 2.18]$

Is $x+y^2 < y$? Maybe.

Domain Tracking

Is $\sqrt{x} < y$? Yes.

Domain Tracking

Is $\sqrt{x} < y$? **Maybe.**

Iterative Graphing Algorithm

- Begin with a solid red display
- Keep a list of uncertain regions
- Evaluate uncertain regions, coloring and subdividing as appropriate

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Pixel Boundaries

- True pixel boundaries may not be floating-point numbers

Example Inequality

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$$\cos 18x + \cos 9\left(x - \sqrt{3}\,y\right) + \cos 9\left(x + \sqrt{3}\,y\right) + 3 < 6\begin{cases} \frac{3}{4} - \frac{1}{15}\sqrt{(x+4)^2 + (y-3)^2} & \text{if} \quad (x+1)^2 + (y-1)^2 < 25 \\ 0.65 + \frac{1}{\pi}\operatorname{Arctan} 6\left[\sqrt{\frac{(x-1)^2}{30} + \frac{(y+1)^2}{9}} - 1\right] & \text{if} \quad (x+1)^2 + (y-1)^2 > 25 \end{cases}$$

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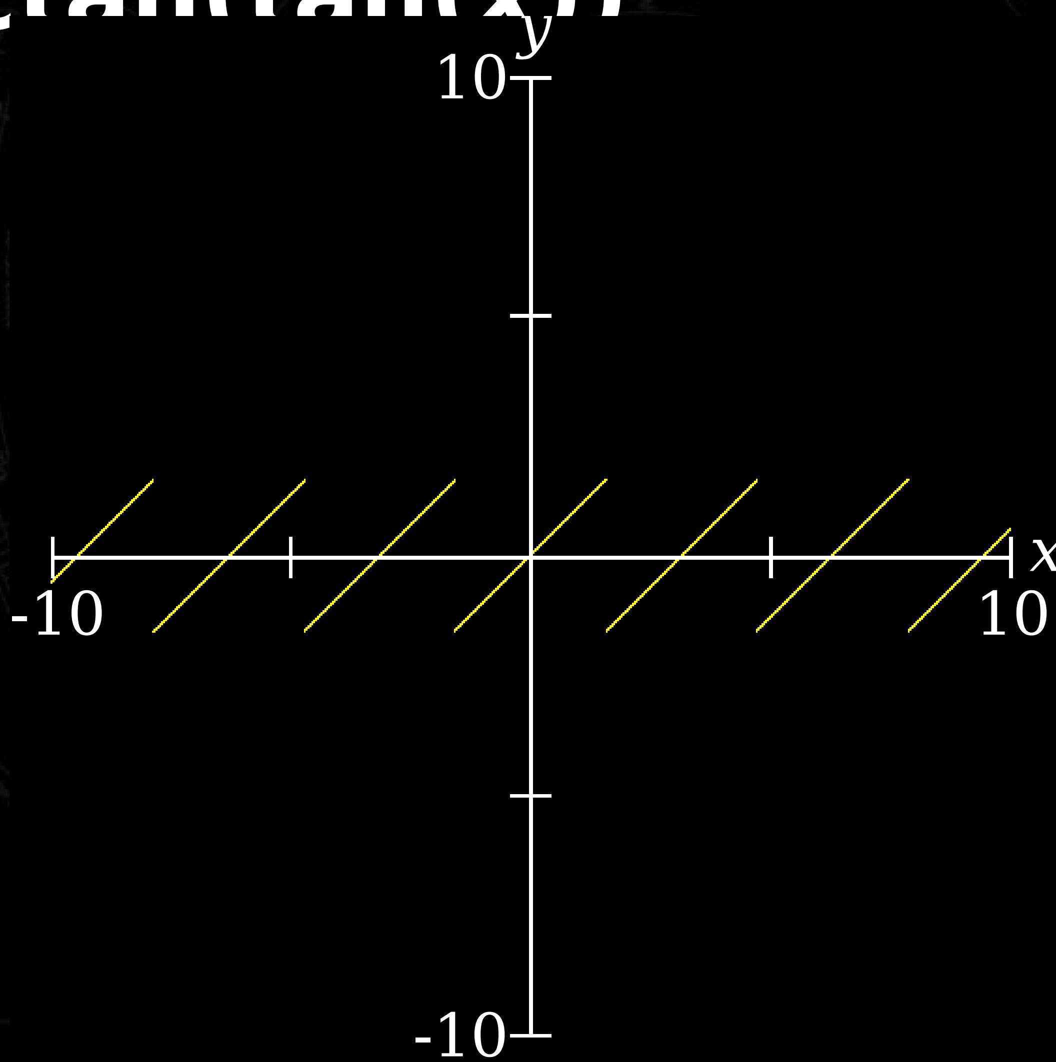

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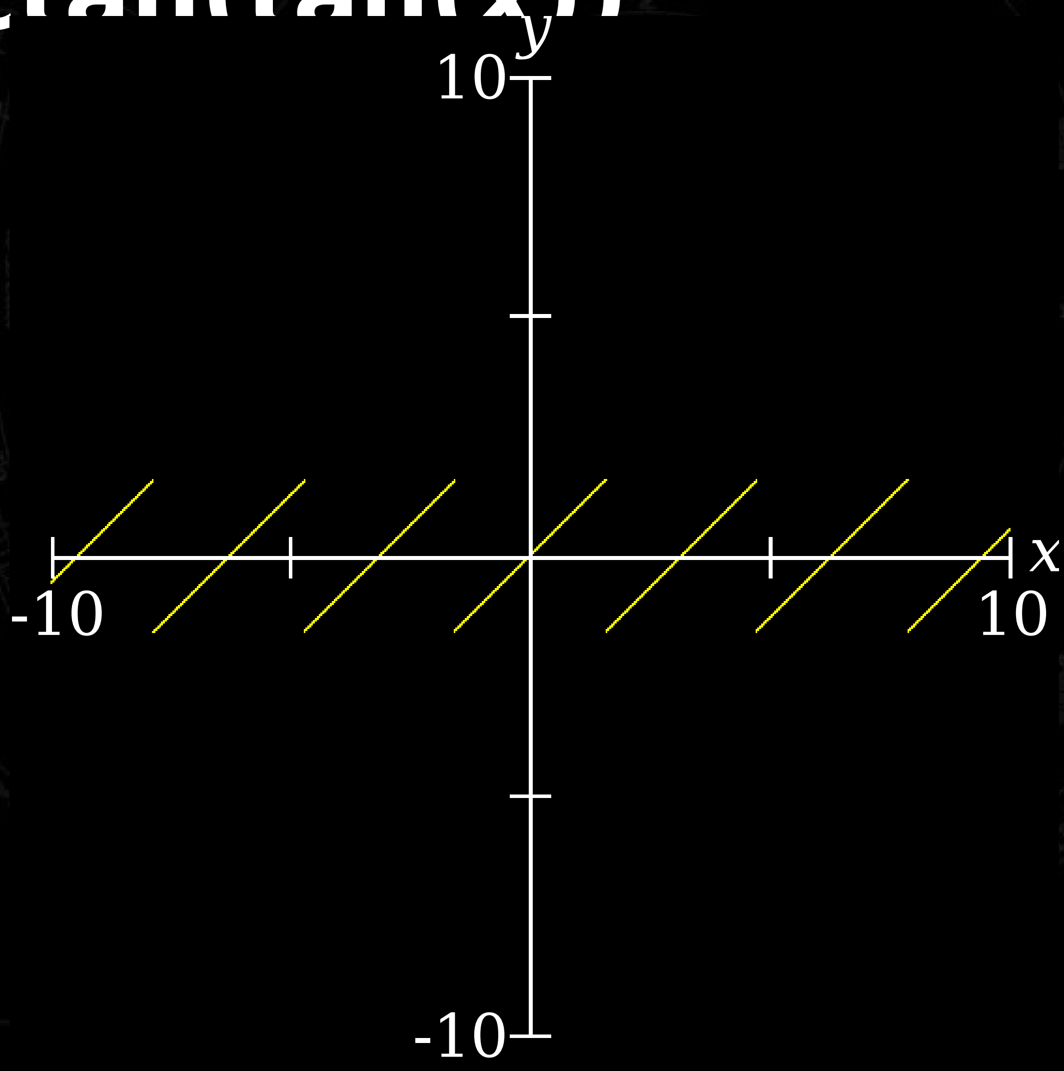
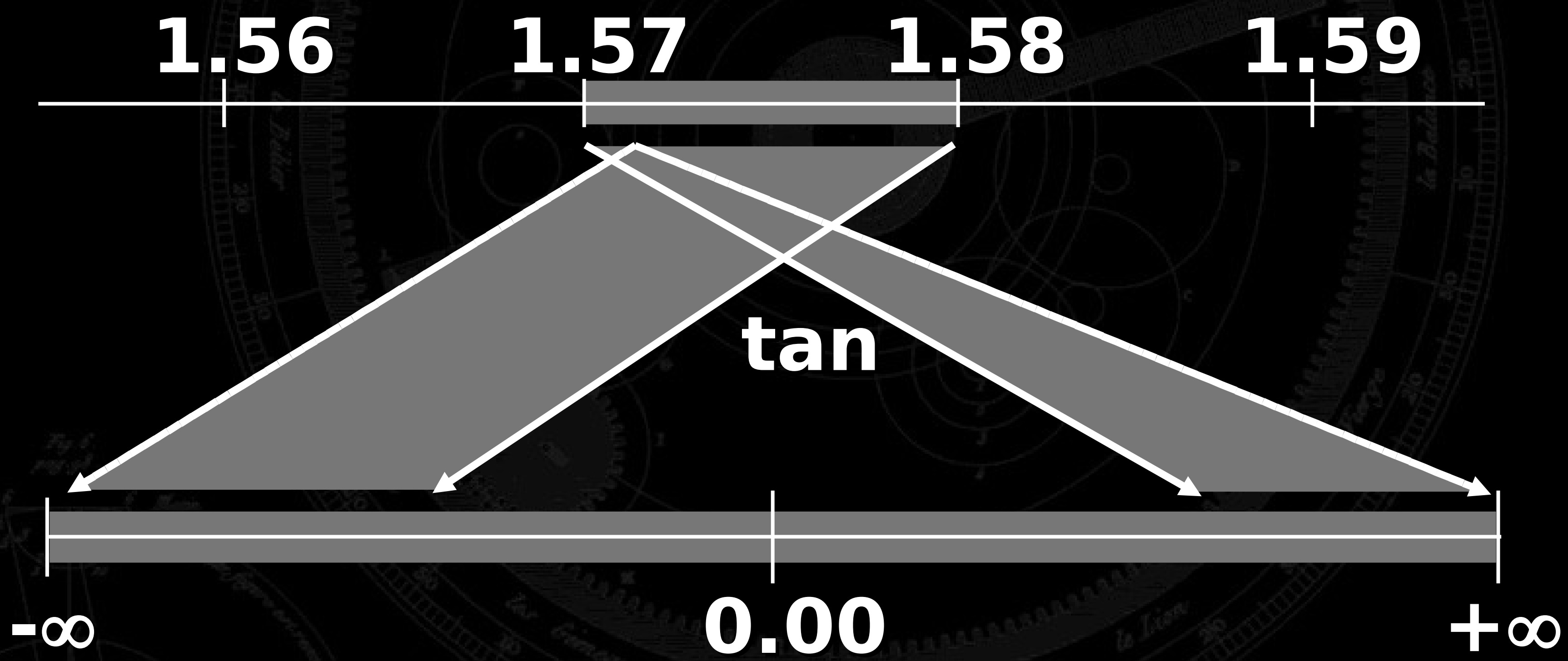
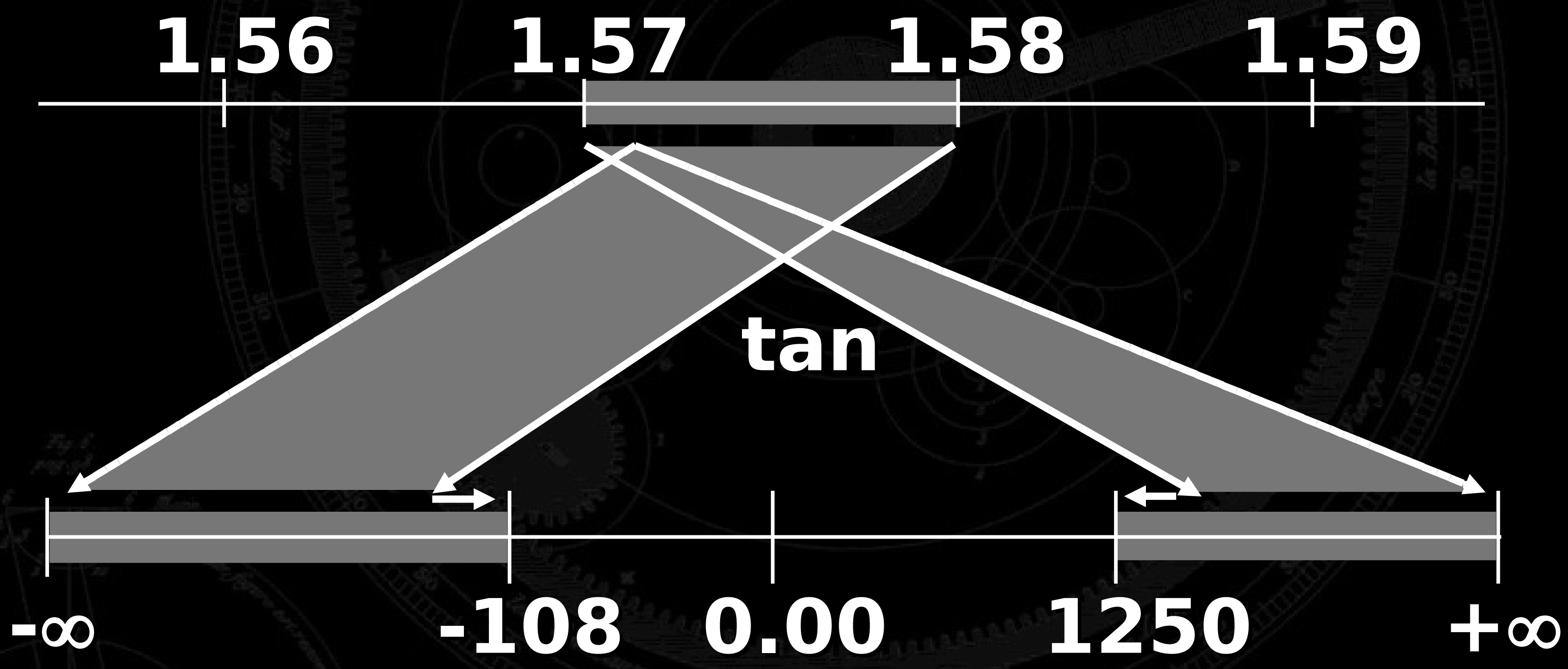


Interval Arithmetic



Interval Sets



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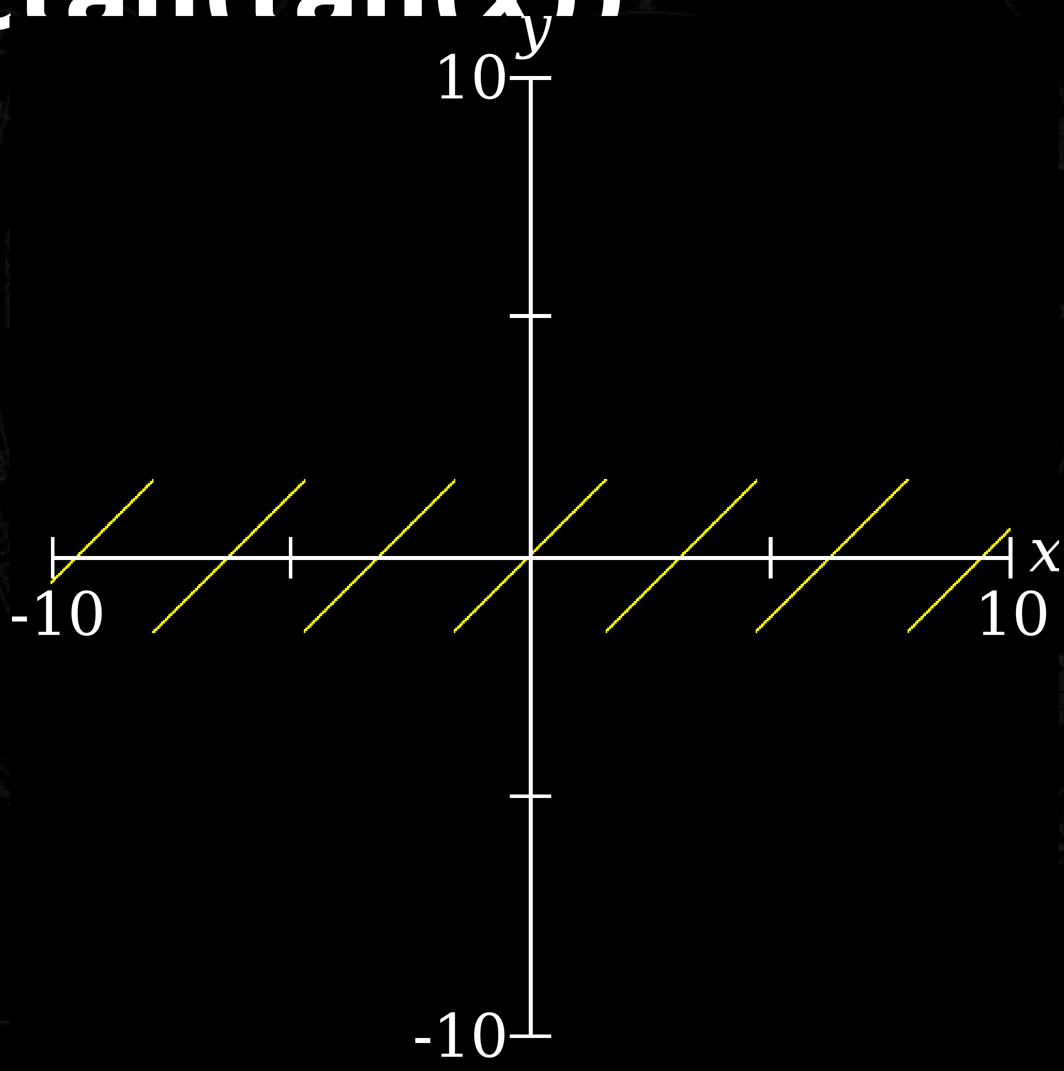
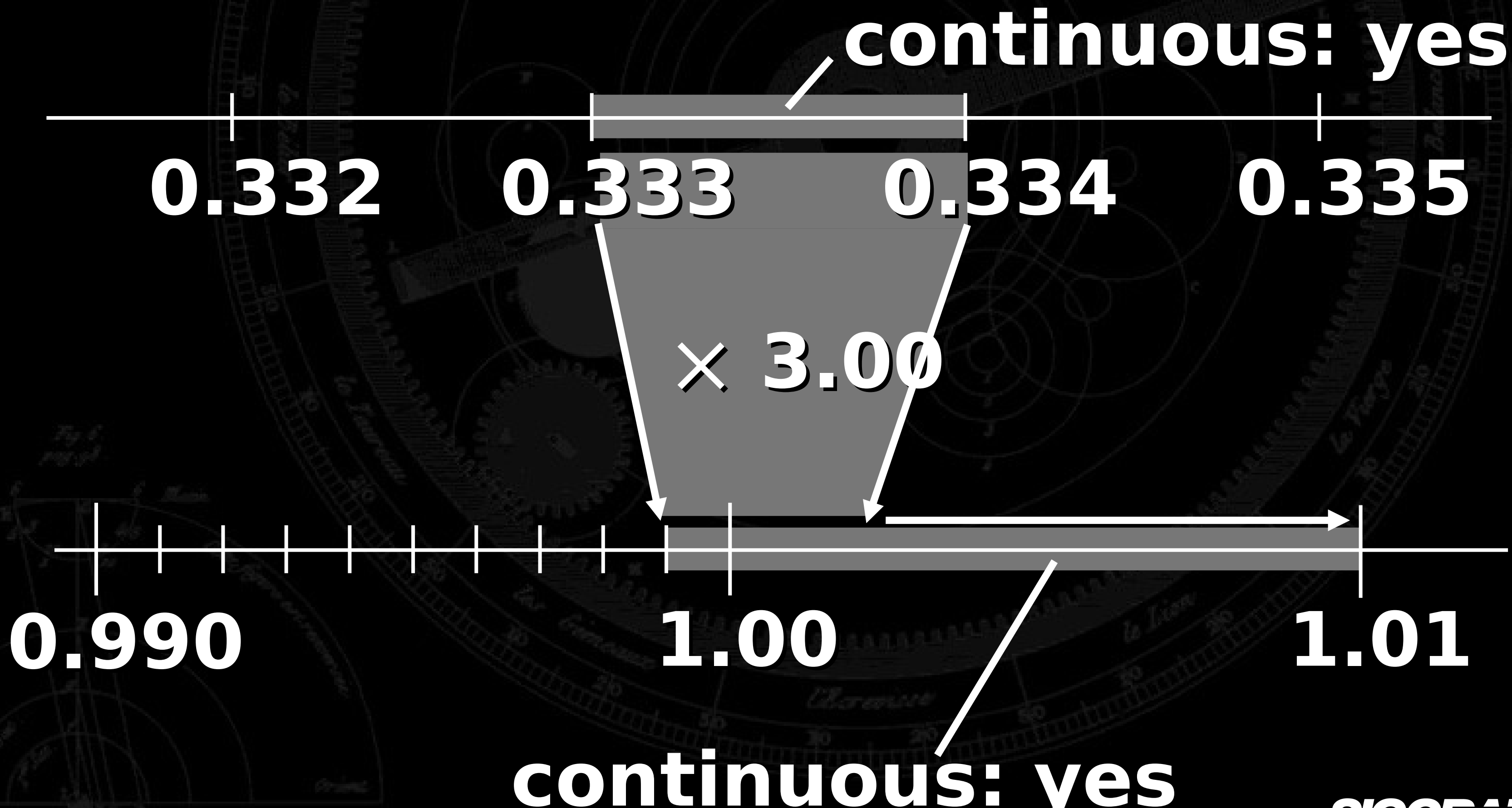


Continuity Tracking



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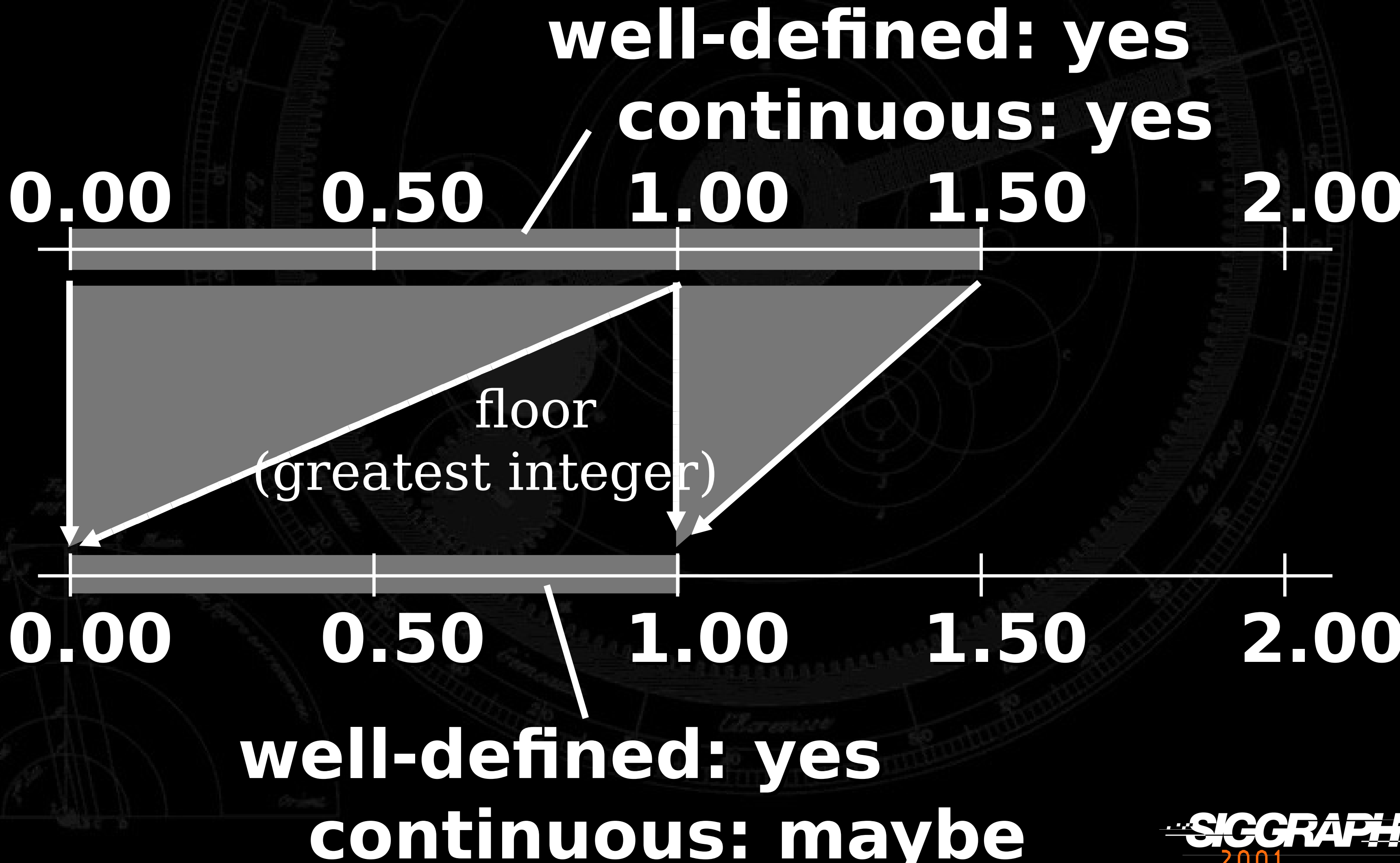
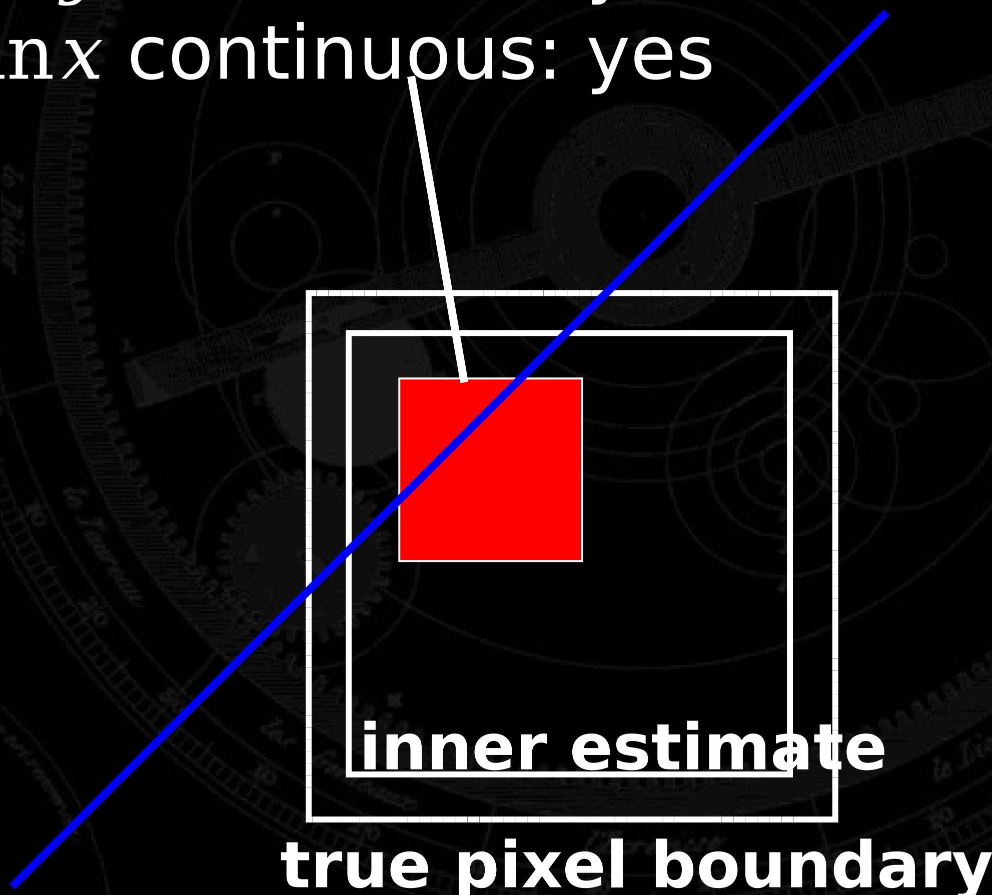


Finding Solutions on Curves

y continuous: yes

$\mathrm{Arctan}\tan x$ continuous: yes



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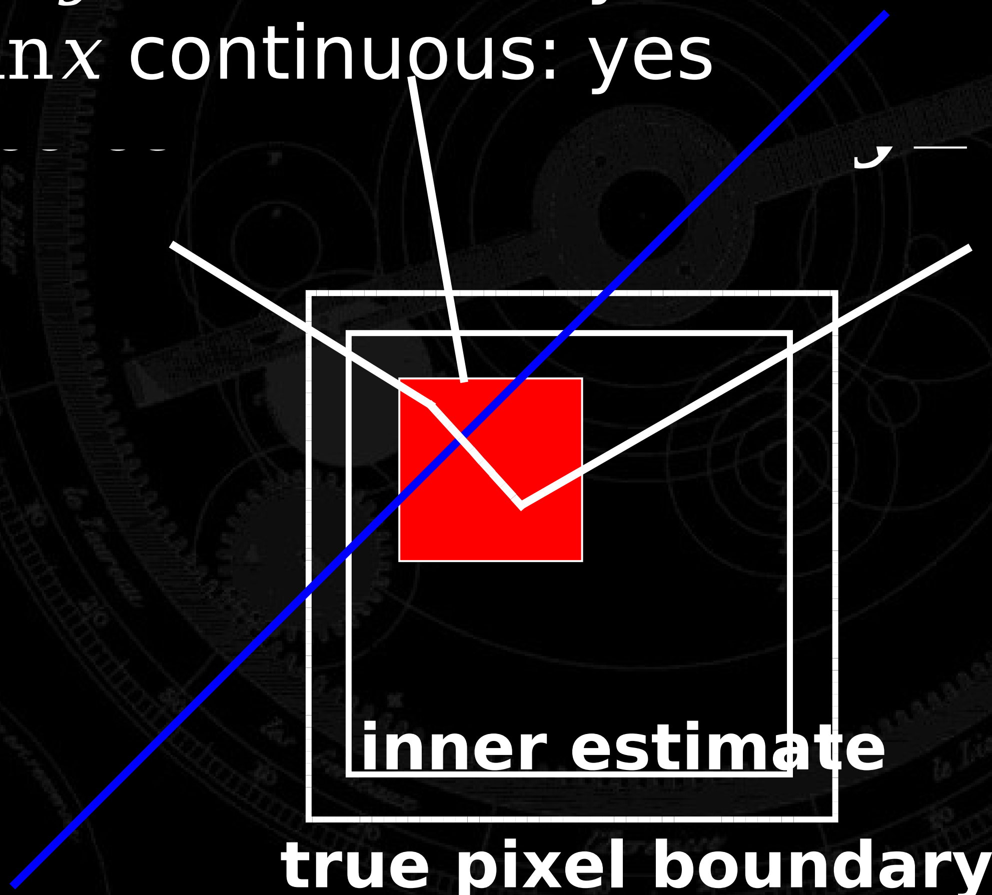

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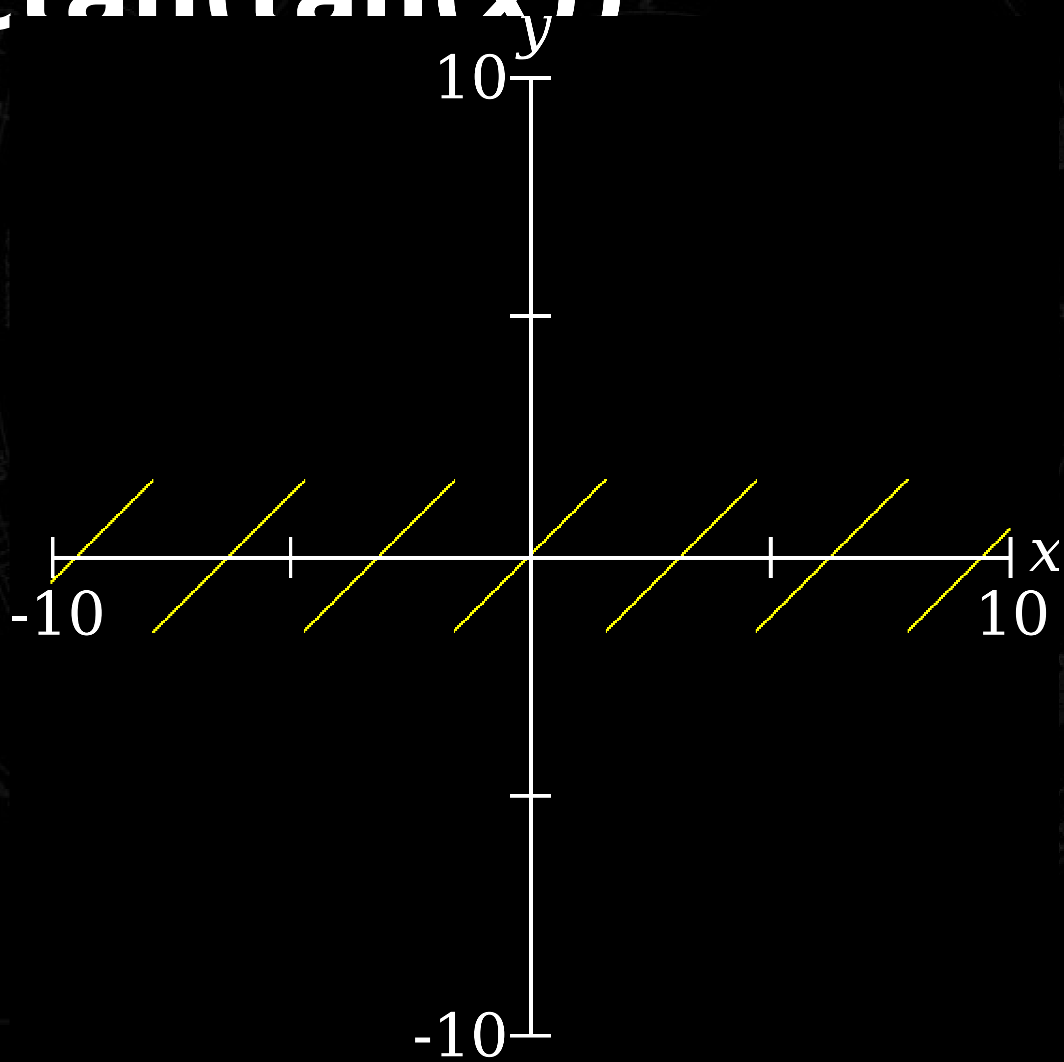

Example Equation

$$|x\cos x + y\sin y| = x\cos y - y\sin x$$

Example Inequality

$$\mathrm{mod}\left(\left\{\begin{array}{lll} 892 & \text{if} & 30\max\left(\left|Y-\frac{1}{2}\right|,\left|X-\frac{1}{2}\right|\right)+\left|Y-\frac{1}{2}\right|-\frac{1}{4}<1 \\ 365 & \text{if} & 30\max\left(\left|Y-\frac{1}{10}\right|,\left|X-\frac{1}{2}\right|\right)+\left|Y-\frac{1}{10}\right|-\frac{1}{4}<1 \\ 941 & \text{if} & 30\max\left(\left|Y-\frac{9}{10}\right|,\left|X-\frac{1}{2}\right|\right)+\left|Y-\frac{9}{10}\right|-\frac{1}{4}<1 \\ 927 & \text{if} & 30\max\left(\left|X-\frac{4}{5}\right|,\left|Y-\frac{7}{10}\right|\right)+\left|X-\frac{4}{5}\right|-\frac{1}{8}<1 \\ 881 & \text{if} & 30\max\left(\left|X-\frac{1}{5}\right|,\left|Y-\frac{7}{10}\right|\right)+\left|X-\frac{1}{5}\right|-\frac{1}{8}<1 \\ 325 & \text{if} & 30\max\left(\left|X-\frac{1}{5}\right|,\left|Y-\frac{3}{10}\right|\right)+\left|X-\frac{1}{5}\right|-\frac{1}{8}<1 \\ 1019 & \text{if} & 30\max\left(\left|X-\frac{4}{5}\right|,\left|Y-\frac{3}{10}\right|\right)+\left|X-\frac{4}{5}\right|-\frac{1}{8}<1 \end{array}\right\}(2)^{-\left\lfloor \mathrm{mod}\left(\frac{\lfloor y\rfloor}{10^{-\lfloor 1.25x\rfloor}},10\right)\right\rfloor},2\right)\geq 1$$

$$X = \mathrm{mod}(x, 0.8) + 0.1$$
$$Y = \mathrm{mod}(y, 1)$$
$$0.8\lfloor -\log y\rfloor < x < 0$$

Example Inequality

$$\operatorname{mod}\left(\left\lfloor \begin{Bmatrix} 892 & \text{if} & 30\max\left(\left|Y-\frac{1}{2}\right|,\left|X-\frac{1}{2}\right|+\left|Y-\frac{1}{2}\right|-\frac{1}{4}\right)<1 \\ 365 & \text{if} & 30\max\left(\left|Y-\frac{1}{10}\right|,\left|X-\frac{1}{2}\right|+\left|Y-\frac{1}{10}\right|-\frac{1}{4}\right)<1 \\ 941 & \text{if} & 30\max\left(\left|Y-\frac{9}{10}\right|,\left|X-\frac{1}{2}\right|+\left|Y-\frac{9}{10}\right|-\frac{1}{4}\right)<1 \\ 927 & \text{if} & 30\max\left(\left|X-\frac{4}{5}\right|,\left|Y-\frac{7}{10}\right|+\left|X-\frac{4}{5}\right|-\frac{1}{8}\right)<1 \\ 881 & \text{if} & 30\max\left(\left|X-\frac{1}{5}\right|,\left|Y-\frac{7}{10}\right|+\left|X-\frac{1}{5}\right|-\frac{1}{8}\right)<1 \\ 325 & \text{if} & 30\max\left(\left|X-\frac{1}{5}\right|,\left|Y-\frac{3}{10}\right|+\left|X-\frac{1}{5}\right|-\frac{1}{8}\right)<1 \\ 1019 & \text{if} & 30\max\left(\left|X-\frac{4}{5}\right|,\left|Y-\frac{3}{10}\right|+\left|X-\frac{4}{5}\right|-\frac{1}{8}\right)<1 \end{Bmatrix} (2)^{-\left\lfloor \operatorname{mod}\left(\frac{\lfloor y\rfloor}{10^{-\lfloor 1.25x\rfloor}},10\right)\right\rfloor}\right\rfloor,2\right)\geq 1$$

$$X=\operatorname{mod}(x,0.8)+0.1$$
$$Y=\operatorname{mod}(y,1)$$
$$0.8\lfloor -\log y\rfloor < x < 0$$

Conclusion

- **Most graphing programs are not reliable**
  - Reliable graphing programs do exist (GrafEq)
- **Red pixels are useful**
- **Be careful when using interval arithmetic**
  - Keeping track of the mathematical properties of evaluated formulae is possible and useful

Future Work

- **Use other colors besides red**
  - Display topological information
- **Tackle a larger class of formulae**
  - integration, differentiation, iteration, ...
- **Animation**
  - visualize role of parameters
- **3D**

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GrafEq:

- www.peda.com/grafeq
- Creative Applications Lab 1PM-2PM Today

Example Inequality

$$\sin(5(\sin x - \sin y)) > \sqrt{|\sin x + \sin y|}$$

Example Equation

$$\operatorname{mod}(\sin x, \cos y) = \operatorname{mod}(\sin y, \cos x)$$

Correct Graph of *y*=Arctan(tan(*x*))